International Journal of Theoretical Physics, Vol. 37,, No. 4,, 1998

Coulomb Blockade and Quantum Fluctuation of a Nondissipative Mesoscopic Capacitance Coupling Circuit with Source

Zhao-xian Yu[1,2] and Ye-hou Liu[2]

Received July 10, 1997

The quantum mechanical effect of electric charge in a nondissipative mesoscopic capacitance coupling circuit is studied and the condition for Coulomb blockade (CCB) is derived. It is pointed out that the CCB is related not only to the junction capacitance, but also to the inductance. The quantum fluctuation of this circuit is also discussed.

1. INTRODUCTION

In recent years, rapid progress in nanometer techniques and nanoelectronics (Srivastava and Widom, 1987; Buot, 1993) has made it possible for the miniaturization of integrated circuits and components to reach atomic dimensions (Garcia, 1992). When the transport dimension reaches a characteristic length, namely when the charge-carrier inelastic coherence length and the charge-carrier confinement dimension approach the Fermi wavelength, the physics of classical devices based on the motion of particles and ensemble averaging is expected to be invalid. The wave nature of electrons, discreteness of energy levels, and sample-specific properties must now be taken into account and quantum mechanical effects should become much more important. Recently, much attention has been paid to the study of mesoscopic physics (Dekker, 1979; Chen *et al.*, 1995, 1996a, b; Li and Chen, 1996; Yu *et al.*, 1997a, b). On the basis of our recent work (Yu *et al.*, 1997a), the present paper studies the quantum mechanical effect of electric charge in a nondissipative mesoscopic capacitance coupling circuit, derives the condition

[1] Shengli Oilfield TV University, Shandong Province 257004, China.
[2] Chongqing Petroleum Advanced Polytechnic College, Chongqing 630042, China.

0020-7748/98/0400-121 7$15.00/0

for Coulomb blockade (CCB), and studies the quantum fluctuation of this circuit.

2. Coulomb Blockade of a Nondissipative Mesoscopic Capacitance Coupling Circuit

The classical Hamiltonian for a classical nondissipative capacitance coupling circuit with source is

$$H = \frac{P_1^2}{2L_1} + \frac{q_1^2}{2C_1} + \frac{(q_1 - q_2)^2}{2C_0} + \frac{P_2^2}{2L_2} + \frac{q_2^2}{2C_2} - q_1\,\mathscr{E}(t) \tag{1}$$

where L_i, C_i, C_0, and q_i ($i = 1, 2$) stand for inductance, capacitances, and charge, respectively. $\mathscr{E}(t)$ is the electromotive force, a function of time. The quantities $p_i = L_i\, dq_i/dt$ and q_i ($i = 1, 2$) are conjugate. If we let

$$[\hat{q}_i, \hat{p}_i] = i\hbar \tag{2}$$

then the quantization of equation (1) can be realized. In the previous work (Yu *et al.*, 1997a) we only considered the continuous electric charge. As a matter of fact, the electric charge is discrete and this must play an important role in the theory of quantized mesoscopic circuits. To take account of the discreteness of electric charge, we impose that the eigenvalues of the self-adjoint operator q_i ($i = 1, 2$) take discrete values, namely

$$\hat{q}_1|q\rangle_1 = nq_e|q\rangle_1, \qquad \hat{q}_2|q\rangle_2 = mq_e|q\rangle_2 \tag{3}$$

where $n, m \in Z$ (set of integers), and $q_e = 1.602 \times 10^{-19}$ C, the elementary electric charge; $|q\rangle_1$ and $|q\rangle_2$ stand for eigenstates of electric charge for circuit 1 and circuit 2, respectively. So we can define two minimum shift operators

$$\hat{Q}_1 = \exp\,(iq_e\hat{p}_1/\hbar), \qquad \hat{Q}_2 = \exp\,(iq_e\hat{p}_2/\hbar) \tag{4}$$

for which the following commutation relations hold:

$$[\hat{q}_1, \hat{Q}_1] = -q_e\hat{Q}_1, \qquad [\hat{q}_1, \hat{Q}_1^+] = q_e\hat{Q}_1^+, \qquad \hat{Q}_1^+\hat{Q}_1 = \hat{Q}_1\hat{Q}_1^+ = 1 \tag{5}$$

$$[\hat{q}_2, \hat{Q}_2] = -q_e\hat{Q}_2, \qquad [\hat{q}_2, \hat{Q}_2^+] = q_e\hat{Q}_2^+, \qquad \hat{Q}_2^+\hat{Q}_2 = \hat{Q}_2\hat{Q}_2^+ = 1 \tag{6}$$

These relations can determine the structure of the whole Fock space. We can also obtain the complete relations, i.e., $\Sigma_{n\varepsilon Z}\,|n\rangle\langle n| = 1$ and $\Sigma_{m\varepsilon Z}|m\rangle\langle m| = 1$. On the other hand, we can also obtain the eigenstate and eigenvalue of the operators p_1 and p_2, respectively.

Similar to the known results (Li and Chen, 1996), we can define the right and left discrete derivative operators as follows:

$$\nabla_{q_e}^{(1)} = (\hat{Q}_1 - 1)/q_e, \qquad \overline{\nabla}_{q_e}^{(1)} = (1 - \hat{Q}_1^+)/q_e \tag{7}$$

$$\nabla_{q_e}^{(2)} = (\hat{Q}_2 - 1)/q_e, \qquad \overline{\nabla}_{q_e}^{(2)} = (1 - \hat{Q}_2^+)/q_e \tag{8}$$

Consequently, one can obtain the self-adjoint "momentum" operators

$$\hat{P}_1 = \frac{\hbar}{2i}(\overline{\nabla}_{q_e}^{(1)} + \nabla_{q_e}^{(1)}) = \frac{\hbar}{2iq_e}(\hat{Q}_1 - \hat{Q}_1^+) \tag{9}$$

$$\hat{P}_2 = \frac{\hbar}{2i}(\nabla_{q_e}^{(2)} + \overline{\nabla}_{q_e}^{(2)}) = \frac{\hbar}{2iq_e}(\hat{Q}_2 - \hat{Q}_2^+) \tag{10}$$

and the free Hamiltonian operators

$$\hat{H}_0^{(1)} = \frac{-\hbar^2}{2}\nabla_{q_e}^{(1)}\overline{\nabla}_{q_e}^{(1)} = \frac{-\hbar^2}{2q_e^2}(\hat{Q}_1 + \hat{Q}_1^+ - 2) \tag{11}$$

$$\hat{H}_0^{(2)} = \frac{-\hbar^2}{2}\nabla_{q_e}^{(2)}\overline{\nabla}_{q_e}^{(2)} = \frac{-\hbar^2}{2q_e^2}(\hat{Q}_2 + \hat{Q}_2^+ - 2) \tag{12}$$

We consider the adiabatic approximation so that $\mathcal{E}(t)$ is considered as a constant $\mathcal{E}$. Regarding the discreteness of electric charge, the Schrödinger equation for a nondissipative mesoscopic capacitance coupling circuit takes the form

$$\left[\frac{-\hbar^2}{2q_e^2L_1}(\hat{Q}_1 + \hat{Q}_1^\dagger - 2) + \frac{\hat{q}_1^2}{2C_{10}} + \frac{-\hbar^2}{2q_e^2L_2}(\hat{Q}_2 + \hat{Q}_2^+ - 2) + \frac{\hat{q}_2^2}{2C_{20}} - \frac{\hat{q}_1\hat{q}_2}{C_0} - \hat{q}_1\mathcal{E}\right]\psi = E\psi \tag{13}$$

where $C_{10} = C_1C_0/(C_1 + C_0)$ and $C_{20} = C_2C_0/(C_2 + C_0)$.

We introduce the following transformations:

$$\begin{bmatrix}\hat{q}_1\\ q_2\end{bmatrix} = \begin{bmatrix}\sqrt{\frac{L_2}{L_1}} & 0\\ 0 & \sqrt{\frac{L_1}{L_2}}\end{bmatrix}\begin{bmatrix}\cos\frac{\varphi}{2} & \sin\frac{\varphi}{2}\\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{bmatrix}\begin{bmatrix}\hat{q}_1'\\ \hat{q}_2'\end{bmatrix} \tag{14}$$

$$\begin{bmatrix}\hat{P}_1\\ \hat{P}_2\end{bmatrix} = \begin{bmatrix}\sqrt{\frac{L_1}{L_2}} & 0\\ 0 & \sqrt{\frac{L_2}{L_1}}\end{bmatrix}\begin{bmatrix}\cos\frac{\varphi}{2} & \sin\frac{\varphi}{2}\\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{bmatrix}\begin{bmatrix}\hat{P}_1'\\ \hat{P}_2'\end{bmatrix} \tag{15}$$

Utilizing equations (14) and (15), we find for equation (13)

$$\left[\frac{-\hbar^2}{2q_e^2L_1}\left\{2\cos\left[\frac{q_e}{\hbar}\sqrt{\frac{L_1}{L_2}}\left(\hat{P}_1'\cos\frac{\varphi}{2}+\hat{P}_2'\sin\frac{\varphi}{2}\right)\right]-2\right\}\right.$$
$$+\frac{-\hbar^2}{2q_e^2L_2}\left\{2\cos\left[\frac{q_e}{\hbar}\sqrt{\frac{L_2}{L_1}}\left(-\hat{P}_1'\sin\frac{\varphi}{2}+P_2'\cos\frac{\varphi}{2}\right)\right]-2\right\}$$
$$\left.+\frac{1}{2\alpha}(\hat{q}_1'-\mathscr{E}A)^2+\frac{1}{2\beta}(\hat{q}_2'-\mathscr{E}\beta)^2+\gamma\hat{q}_1'\hat{q}_2'\right]\tilde{\psi}=\tilde{E}\tilde{\psi} \tag{16}$$

where

$$\alpha^{-1}=\frac{1}{C_{10}}\left(\frac{L_2}{L_1}\right)\cos^2\frac{\varphi}{2}+\frac{1}{C_{20}}\left(\frac{L_1}{L_2}\right)\sin^2\frac{\varphi}{2}+\frac{\sin\varphi}{C_0} \tag{17}$$

$$\beta^{-1}=\frac{1}{C_{10}}\left(\frac{L_2}{L_1}\right)\sin^2\frac{\varphi}{2}+\frac{1}{C_{20}}\left(\frac{L_1}{L_2}\right)\cos^2\frac{\varphi}{2}-\frac{\sin\varphi}{C_0} \tag{18}$$

$$\gamma=\frac{1}{2C_{10}}\left(\frac{L_2}{L_1}\right)\sin\varphi-\frac{1}{2C_{20}}\left(\frac{L_1}{L_2}\right)\sin\varphi-\frac{\cos\varphi}{C_0} \tag{19}$$

$$A=\alpha\sqrt{\frac{L_2}{L_1}}\cos\frac{\varphi}{2},\qquad B=\beta\sqrt{\frac{L_2}{L_1}}\sin\frac{\varphi}{2} \tag{20}$$

In order to eliminate the cross term in equation (16), let $\gamma=0$; one have

$$\tan\varphi=\frac{2}{C_0}\left[\left(\frac{1}{C_0}+\frac{1}{C_1}\right)\left(\frac{L_2}{L_1}\right)-\left(\frac{1}{C_0}+\frac{1}{C_2}\right)\left(\frac{L_1}{L_2}\right)\right]^{-1} \tag{21}$$

Due to the discreteness of electric charge, equation (3) must hold, i.e.,

$$\mathscr{E}=\frac{-nq_e}{\alpha\sqrt{L_2/L_1}\cos(\varphi/2)},\qquad \mathscr{E}=\frac{-mq_e}{\beta\sqrt{L_2/L_1}\sin(\varphi/2)}\qquad (n,m\in Z) \tag{22}$$

We have

$$\mathscr{E}=\frac{-mnq_e}{m\cos(\varphi/2)+n\sin(\varphi/2)}\left[\left(\frac{1}{C_1}+\frac{1}{C_0}\right)\sqrt{\frac{L_2}{L_1}}+\left(\frac{1}{C_2}+\frac{1}{C_0}\right)\left(\frac{L_1}{L_2}\right)^{3/2}\right] \tag{23}$$

$$\frac{m}{n}=\frac{\beta}{\alpha}\tan\frac{\varphi}{2} \tag{24}$$

Equations (23) and (24) are the CCB of a nondissipative mesoscopic capacitance coupling circuit. We find that the adiabatic approximation electromotive force $\mathscr{E}$ only takes discrete values, given by equation (24). On the other hand, we can also find that the CCB is related not only to the junction capacitances, but also to the inductances.

3. QUANTUM FLUCTUATION OF A NONDISSIPATIVE MESOSCOPIC CAPACITANCE COUPLING CIRCUIT

According to the relations (Li and Chen, 1996)

$$\langle p_i' | \hat{Q}_i + \hat{Q}_i^+ - 2 | p_i \rangle = \frac{4\pi\hbar}{q_e}\left[\cos\left(\frac{q_e}{\hbar} p_i\right) - 1\right]\delta(p_i - p_i') \tag{25}$$

$$\langle p_i' | \hat{q}_i^2 | p_i \rangle = \frac{-2\pi\hbar^3}{q_e}\frac{\partial^2}{\partial p_i^2}\delta(p_i - p_i') \tag{26}$$

the formula for expressing equation (16) in the p representation is

$$\begin{aligned}&\left[\frac{-\hbar^2}{q_e^2 L_1}\left\{\cos\left[\frac{q_e}{\hbar}\sqrt{\frac{L_1}{L_2}}\left(p_1' \cos\frac{\varphi}{2} + p_2' \sin\frac{\varphi}{2}\right)\right] - 1\right\}\right.\\ &\quad + \frac{-\hbar^2}{q_e^2 L_2}\left\{\cos\left[\frac{q_e}{\hbar}\sqrt{\frac{L_2}{L_1}}\left(-p_1' \sin\frac{\varphi}{2} + p_2' \cos\frac{\varphi}{2}\right)\right] - 1\right\}\\ &\quad \left. + \frac{-\hbar^2}{2\alpha}\frac{\partial^2}{\partial p_1'^2} + \frac{-\hbar^2}{2\beta}\frac{\partial^2}{\partial p_2'^2}\right]\tilde{\psi}\left(p_1', p_2'\right) = \mathrm{E}\tilde{\psi}\left(p_1', p_2'\right)\end{aligned} \tag{27}$$

where we have adopted $\mathscr{E} = 0$ for simplicity.

It is difficult to obtain the exact solution of equation (27). We now make the following approximate discussion. Choosing the proper values of L_i and C_i so as to make sin $(\varphi/2) \sim 0$ (in this case, we do not take $\cos(\varphi/2) = 1$], we find that equation (27) is equivalent to the two Mathieu equations

$$\left\{\frac{-\hbar^2}{2\alpha}\frac{\partial^2}{\partial p_1'^2} + \frac{-\hbar^2}{q_e^2 L_1}\left[\cos\left(\frac{q_e}{\hbar}\sqrt{\frac{L_1}{L_2}}\cos\frac{\varphi}{2}p_1'\right) - 1\right]\right\}\tilde{\psi}(p_1') = \tilde{E}_1\tilde{\psi}(p_1') \tag{28}$$

$$\left\{\frac{-\hbar^2}{\partial\beta}\frac{\partial^2}{\partial p_2'^2} + \frac{-\hbar^2}{q_e^2 L_2}\left[\cos\left(\frac{q_e}{\hbar}\sqrt{\frac{L_2}{L_1}}\cos\frac{\varphi}{2}p_2'\right) - 1\right]\right\}\tilde{\psi}(p_2') = \tilde{E}_2\tilde{\psi}(p_2') \tag{29}$$

In terms of the conventional notations (Wang and Guo, 1965; Gradshteyn and Ryzhik, 1980), the solution of equation (28) is

$$\tilde{\psi}_{l_1}^+(p_1') = ce_{l_1}\left(\frac{\pi}{2} - \frac{q_e}{2\hbar}\sqrt{\frac{L_1}{L_2}}\cos\frac{\varphi}{2}p_1', \xi_1\right) \tag{30}$$

or

$$\tilde{\psi}^{-}_{l_1+1}(p'_1) = se_{l_1+1}\left(\frac{\pi}{2} - \frac{q_e}{2\hbar}\sqrt{\frac{L_1}{L_2}}\cos\frac{\varphi}{2}p'_1, \xi_1\right) \tag{31}$$

and the solution of equation (29) is

$$\tilde{\psi}^{+}_{l_2}(p'_2) = ce_{l_2}\left(\frac{\pi}{2} - \frac{q_e}{2\hbar}\sqrt{\frac{L_2}{L_1}}\cos\frac{\varphi}{2}p'_2, \xi_2\right) \tag{32}$$

or

$$\tilde{\psi}^{-}_{l_2+1}(p'_2) = se_{l_2+1}\left(\frac{\pi}{2} - \frac{q_e}{2\hbar}\sqrt{\frac{L_2}{L_1}}\cos\frac{\varphi}{2}p'_2, \xi_2\right) \tag{33}$$

where "+" and "−" denote the even and odd parity solutions, respectively, and we have quantum numbers l_1, $l_2 = 0, 1, 2, \ldots$; $\xi_1 = (2\hbar/q_e^2)^2\alpha/L_1$ and $\xi_2 = (2\hbar/q_e^2)^2\beta/L_2$. Here $ce(z, \xi)$ and $se(z, \xi)$ are periodic Mathieu functions. In this case, there exist infinitely many eigenvalues $\{a_1\}$ and $\{b_1\}$ that are not identically equal to zero. Then the energy spectrum is expressed in terms of the eigenvalues a_1, b_1 of the Mathieu equation,

$$\tilde{E}^{+}_{l_1} = \frac{q_e^2}{8\alpha}a_{l_1}(\xi_1) + \frac{\hbar^2}{q_e^2L_1}, \qquad \tilde{E}^{-}_{l_1+1} = \frac{q_e^2}{8\alpha}b_{l_1+1}(\xi_1) + \frac{\hbar^2}{q_e^2L_1} \tag{34}$$

$$\tilde{E}^{+}_{l_2} = \frac{q_e^2}{8\beta}a_{l_2}(\xi_2) + \frac{\hbar^2}{q_e^2L_2}, \qquad \tilde{E}^{-}_{l_2+1} = \frac{q_e^2}{8\beta}b_{l_2+1}(\xi_2) + \frac{\hbar^2}{q_e^2L_2} \tag{35}$$

In consideration of the conditions $\xi_1 << 1$ and $\xi_2 << 1$, the Mathieu equation can be solved by the WKB method. The self-adjoint operators, after transformation by equation (15), take the forms

$$\hat{P}'_1 = \frac{\hbar}{q_e}\sin\left[\frac{q_e}{\hbar}\sqrt{\frac{L_1}{L_2}}\cos\frac{\varphi}{2}p'_1\right], \qquad \hat{P}'_2 = \frac{\hbar}{q_e}\sin\left[\frac{q_e}{\hbar}\sqrt{\frac{L_2}{L_1}}\cos\frac{\varphi}{2}p'_2\right] \tag{36}$$

We have

$$\langle ce_{l_1}|\hat{P}'_1|ce_{l_1}\rangle = \langle se_{l_1+1}|P'_1|se_{l_1+1}\rangle = 0 \tag{37}$$

$$\langle ce_{l_2}|\hat{P}'_2|ce_{l_2}\rangle = \langle se_{l_2+1}|\hat{P}'_2|se_{l_2+1}\rangle = 0 \tag{38}$$

Equations (37) and (38) indicate that the average value of the current is zero whether this circuit is in the ground state or in an excited state. We now calculate the fluctuation of electric current (apart from a factor $1/L$) for the ground state:

$$\langle ce_0|\hat{P}_1'^2|ce_0\rangle = \frac{1}{2}\left(\frac{\hbar}{q_e}\right)^2\left[1 - \frac{3}{2}\left(\frac{\hbar^2\alpha}{q_e^4 L_1}\right)^2\frac{1}{\cos(\varphi/2)}\sqrt{\frac{L_2}{L_1}} + \ldots\right] \tag{39}$$

$$\langle ce_0|\hat{P}_2'^2|ce_0\rangle = \frac{1}{2}\left(\frac{\hbar}{q_e}\right)^2\left[1 - \frac{3}{2}\left(\frac{\hbar^2\beta}{q_e^4 L_2}\right)^2\frac{1}{\cos(\varphi/2)}\sqrt{\frac{L_1}{L_2}} + \ldots\right] \tag{40}$$

Therefore we conclude that there exist current quantum zero-point fluctuations, and the fluctuations between circuit 1 and circuit 2 are correlated. This conclusion will be useful in the design of nanometer electric circuits.

REFERENCES

Buot, F. A. (1993). *Physics Reports*, **234**, 73.

Chen, B., Gao, S. E., and Jiao, Z. K. (1995). *Acta Physica Sinica*, **44**, 1480 [in Chinese].

Chen, B., Fang, H., Jiao, Z. K., and Zhang, Q. R. (1996a). *Chinese Science Bulletin*, **41**, 1170 [in Chinese].

Chen, B., Li, Y. Q., Sa, J., and Zhang, Q. R. (1996b). *Chinese Science Bulletin*, **41**, 1275 [in Chinese].

Dekker, H. (1979). *Physica*, **95A**, 311.

Garcia, R. G. (1992). *Applied Physics Letters*, **60**, 1960.

Gradshteyn, I. S., and Ryzhik, I. M. (1980). *Tables of Integrals, Series, and Products*, Academic Press, New York.

Li, Y. Q., and Chen, B. (1996). *Physical Review B*, **53**, 4027.

Srivastava, Y., and Widom, A. (1987). *Physics Reports*, **148**, 1.

Wang, Z. X., and Guo, D. R. (1965). *Introduction to Special Functions*, Science Press, Beijing [in Chinese].

Yu, Z. X., Zhang, D. X., and Liu, Y. H. (1997a). *International Journal of Theoretical Physics*, **36**, 1965.

Yu, Z. X., Zhang, D. X., and Liu, Y. H. (1997b). *Acta Physica Sinica (Overseas Edition)*, **6**, 522.